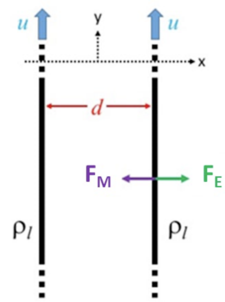
Problem 1: Lorentz Force

Consider two long straight conductors (wires consisting of magnetic material, with relative permeability of $\mu_r = 3$) with electric line charge densities ρ_l (Coulomb/m) as shown in the figure below. The conductors are a distance d apart and the line charges are moving at a constant speed u in the $+\hat{y}$ direction. Obtain the required speed, u, such that the magnetic attraction force ($\overrightarrow{F_m}$) and electric repulsion force ($\overrightarrow{F_e}$) on the two line charge densities are exactly in balance. (To solve for the required speed, begin by obtaining the electric field and magnetic field and their associated forces on each line charge.)



Constant velocity of charges $\longrightarrow$ constant current $\Longrightarrow$ Magnetostatics
Ampere's Law:

$$\oint \overrightarrow{H} \cdot d\overrightarrow{l} = I = \rho_l u$$

$$\overrightarrow{H} = \frac{I}{2\pi d}\hat{\phi}$$

Magnetic Flux density due to the infinite wire with current I:

$$\overrightarrow{B} = \frac{\mu I}{2\pi d}\hat{\phi}$$

(This is evaluated at the 2nd wire due to the $\overrightarrow{B}$ from the 1st wire.)
Now, find the magnetic force per unit length:

$$\begin{aligned}
\overrightarrow{F_m} &= I\int d\overrightarrow{l} \times \overrightarrow{B} \\
&= I\int_{z=0}^{l} (dz\hat{z}) \times (\frac{\mu I}{2\pi d}\hat{\phi}) \\
&= I\frac{\mu I}{2\pi d} l(-\hat{r}) \\
\frac{\overrightarrow{F_m}}{l} &= \frac{\mu I^2}{2\pi d}(-\hat{r})
\end{aligned}$$

(This is the magnetic force per unit length at wire 2 due to wire 1)

We would expect the magnetic forces to attract, whereas the the electric force repels. The magnetic force created by the 1st wire and exerted on the 2nd wire being in the $-\hat{r}$ direction is consistent with this understanding.

Now for the electric force:

$$\oint \overrightarrow{D} \cdot d\overrightarrow{s} = Q_{enc} = \rho_l l$$

(This is the charge enclosed for a segment of length l)

Due to the cylindrical symmetry

$$\overrightarrow{E} = E_r \hat{r}$$

$$\epsilon_0 E_r \cdot 2\pi d l = \rho_l l$$

$$E_r = \frac{\rho_l}{2\pi d \epsilon_0} \Longrightarrow \overrightarrow{F_E} = Q\overrightarrow{E} = \rho_l l \frac{\rho_l}{2\pi d \epsilon_0} \hat{r}$$

$$\frac{\overrightarrow{F_E}}{l} = \frac{\rho_l^2}{2\pi d \epsilon_0} \hat{r}$$

Therefore, since the forces are equal and opposite, we can equate them and solve for u:

$$\left| \frac{F_E}{l} \right| = \left| \frac{F_m}{l} \right|$$

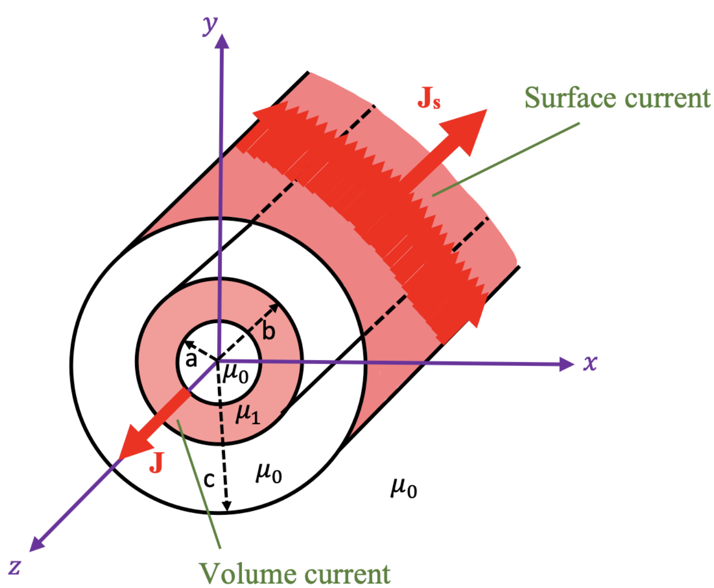
$$\frac{\rho_l^2}{2\pi d \epsilon_0} = \frac{\mu I^2}{2\pi d} \quad \text{where } I = \rho_l u$$

$$\frac{\rho_l^2}{2\pi d \epsilon_0} = \frac{\mu \rho_l^2 u^2}{2\pi d} \quad \text{where } \mu = 3\mu_0$$

$$u = \frac{1}{\sqrt{3\mu_0 \epsilon_0}} = \frac{c}{\sqrt{3}}$$

Problem 2: Ampere's Law

A coaxial system has a hollow inner conductive cylinder of inner radius a and outer radius b along with an outer conductor shell of radius c shown in the figure below. The space outside the coaxial system, between the outer shell and the inner conductor, and inside the hollow conductor is filled with air (ϵ_0, μ_0) and the inner solid conductor has magnetic permeability μ_1.
The outer conductor carries uniform surface density $\overrightarrow{J_S}$ and the total current flowing in the outer conductor shell ($r = b$) is I and in the -$\hat{z}$ direction. The inner hollow cylinder ($a < r < b$) carries volume current density $\overrightarrow{J}$ and the total current flowing inside the inner cylinder is uniform current I in the $+\hat{z}$ direction. Find the magnetic field $\overrightarrow{H}$ in the four regions:



Note: We have cylindrical symmetry with current flowing in the the conductors, so our magnetic field will have the form

$$\overrightarrow{H} = H_\phi(r)\hat{\phi}$$

a **Region 1:** $0 < r < a$

$$\oint \overrightarrow{H} \cdot d\overrightarrow{l} = I_{enc} = 0$$

$$\overrightarrow{H} = 0 \qquad 0 < r < a$$

b **Region 2:** $a \le r \le b$

Due to the <u>uniform</u> current density

$$\vec{J} = \frac{I}{\pi(b^2 - a^2)}\hat{z}$$

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} = \int_{r=a}^{r} \int_{\phi=0}^{2\pi} \vec{J} \cdot d\vec{s}$$

$$H_\phi \cdot 2\pi r = \frac{I}{\pi(b^2 - a^2)}\pi(r^2 - a^2) = \frac{I(r^2 - a^2)}{(b^2 - a^2)}$$

$$H_\phi = \frac{I(r^2 - a^2)}{2\pi r(b^2 - a^2)}$$

$$\vec{H} = H_\phi(r)\hat{\phi} = \frac{I(r^2 - a^2)}{2\pi r(b^2 - a^2)}\hat{\phi} \qquad a < r < b$$

c **Region 3:** $b \le r \le c$

The enclosed current in this region is given as I (total current flowing through the inner conductor in the $+\hat{z}$ direction).

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} = I = \int_{r=0}^{r} \int_{\phi=0}^{2\pi} \vec{J} \cdot d\vec{s}$$

$$H_\phi = \frac{I}{2\pi r}$$

$$\vec{H} = H_\phi(r)\hat{\phi} = \frac{I}{2\pi r}\hat{\phi} \qquad b \le r \le c$$

d **Region 4:** $r > c$

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} = I + (-I) = 0$$

$$\vec{H} = 0 \qquad r > c$$